Vector calculus

I. Prove that $\nabla^2(r^n \vec{r}) = n(n+3) r^{n-2} \vec{r}$

Soln Let $\vec{F} = r^n \vec{r}$

$\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow r^n \vec{r} = r^n (x\vec{i} + y\vec{j} + z\vec{k})$

$\Rightarrow \vec{F} = r^n \vec{r} = r^n x \vec{i} + r^n y \vec{j} + r^n z \vec{k}$

$\Rightarrow \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ (say) —— (1)

where $F_1 = r^n x$, $F_2 = r^n y$, $F_3 = r^n z$

Now $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$\Rightarrow \nabla \cdot \nabla = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$

$\Rightarrow \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ —— (2)

from (1) and (2),

$\Rightarrow \nabla^2 \vec{F} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$

$\Rightarrow \nabla^2 \vec{F} = \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\vec{i} + \left(\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}\right)\vec{j} + \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right)\vec{k}$

—— (3)

Now, $F_1 = r^n x \Rightarrow \frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(r^n x)$

$$\Rightarrow \frac{\partial F_1}{\partial x} = n r^{n-1} \frac{\partial r}{\partial x} x + r^n = n r^{n-1} \cdot x \cdot \frac{x}{r} + r^n$$

$$\Rightarrow \frac{\partial F_1}{\partial x} = n r^{n-2} x^2 + r^n$$

$$\Rightarrow \frac{\partial^2 F_1}{\partial x^2} = n(n-2) r^{n-3} x^2 \cdot \frac{\partial r}{\partial x} + n r^{n-2} \cdot 2x + n r^{n-1} \cdot \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial^2 F_1}{\partial x^2} = n(n-2) r^{n-3} x^2 \cdot \frac{x}{r} + n r^{n-2} \cdot 2x + n r^{n-1} \cdot \frac{x}{r}$$

$$= n(n-2) r^{n-4} x^3 + 2nx r^{n-2} + nx r^{n-2}$$

$$\Rightarrow \frac{\partial^2 F_1}{\partial x^2} = n(n-2) r^{n-4} x^3 + 3nx r^{n-2} \quad (4)$$

$$= nx r^{n-2} \left(3 + (n-2) \frac{x^2}{r^2}\right)$$

Again $\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(r^n x) = xnr^{n-1}\frac{\partial r}{\partial y}$

$$= xnr^{n-1}\frac{y}{r} = nyxr^{n-2}$$

$$\Rightarrow \frac{\partial^2 F_1}{\partial y^2} = \frac{\partial}{\partial y}\left(xnyr^{n-2}\right) = nx\left(r^{n-2} + y(n-2)r^{n-3}\frac{\partial r}{\partial y}\right)$$

$$= xn\left(r^{n-2} + (n-2)yr^{n-3}\cdot\frac{y}{r}\right)$$

$$= xn\left(r^{n-2} + (n-2)y^2r^{n-4}\right)$$

$$\Rightarrow \frac{\partial^2 F_1}{\partial y^2} = nr^{n-2}x\left[1+(n-2)\frac{y^2}{r^2}\right] \quad \text{—(5)}$$

Again $\frac{\partial F_1}{\partial z} = \frac{\partial}{\partial z}(r^n x) = xnr^{n-1}\frac{\partial r}{\partial z}$

$$= xnr^{n-1}\frac{z}{r} = nxr^{n-2}z$$

$$\Rightarrow \frac{\partial^2 F_1}{\partial z^2} = \frac{\partial}{\partial z}\left(nxr^{n-2}z\right) = nx\left((n-2)r^{n-3}\frac{\partial r}{\partial z}\cdot z + r^{n-2}\right)$$

$$= nx\left[(n-2)r^{n-3}\frac{z}{r}z + r^{n-2}\right]$$

$$\Rightarrow \frac{\partial^2 F_1}{\partial z^2} = nxr^{n-2}\left[(n-2)\frac{z^2}{r^2} + 1\right] \quad \text{—(6)}$$

Adding (4), (5) and (6), we've

$$\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}$$

$$= n x r^{n-2}\left[3+(n-2)\frac{x^2}{r^2}+1+(n-2)\frac{y^2}{r^2} + (n-2)\frac{z^2}{r^2}+1\right]$$

$$= n x r^{n-2}\left[5+(n-2)\frac{(x^2+y^2+z^2)}{r^2}\right]$$

$$= n x r^{n-2}\left(5+(n-2)\cdot\frac{r^2}{r^2}\right) = n x r^{n-2}(n+3)$$

$$\Rightarrow \frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2} = n(n+3)\, x\, r^{n-2} \quad \text{— (7)}$$

Similarly $\frac{\partial^2 F_2}{\partial x^2}+\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_2}{\partial z^2} = n(n+3)\, y\, r^{n-2}$ (8) and

$$\frac{\partial^2 F_3}{\partial x^2}+\frac{\partial^2 F_3}{\partial y^2}+\frac{\partial^2 F_3}{\partial z^2} = n(n+3)\, z\, r^{n-2} \quad (9)$$

Putting the values from (7), (8) and (9) in eq (3), we have

$$\therefore \text{LHS} = \nabla^2 \vec{F} = \nabla^2 (r^n \vec{r})$$

$$= n(n+3)\, r^{n-2} (x\hat{i}+y\hat{j}+z\hat{k})$$

$$= n(n+3)\, r^{n-2}\, \vec{r} = \text{RHS}$$